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MORE ON THE AR PROPERTY AND CHAIN CONDITIONS IN GROUP RINGS

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ABSTRACT
We investigate when the augmentation ideal of a group algebra kG has the AR property in the cases of G a locally nilpotent or hyperabelian group.

1. Introduction

Let k be a field of characteristic $p \geqq 0$ and G a group. In an earlier paper [7] we showed that if the augmentation ideal $\mathfrak{g}_k$ of the group algebra kG satisfies the AR property then the ring kG and the group G satisfy certain chain conditions, and using these we were able to characterize those hypercentral groups G such that $\mathfrak{g}_k$ has the AR property (see [7], theorem C′). In this paper, which is really just a rather long postscript to [7], we try to characterize those locally nilpotent or hyperabelian groups G such that $\mathfrak{g}_k$ has the AR property.

The notation used in this paper is taken from [7] with one exception. If p is a prime or zero then a group G will be said to satisfy Max-sn$_p^*$ if for any ascending chain of subgroups $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ such that H_i is normal in H_{i+1} for each $i \geqq 1$ there exists a positive integer n_0 such that H_{n+1}/H_n is a locally finite-p' group for all $n \geqq n_0$. In the case $p = 0$ we shall understand by a $0'$-group an arbitrary group. This definition is clearly stronger than the earlier one, so as our first result we have:

LEMMA A. *Let k be a field of characteristic $p \geqq 0$ and G a group such that $\mathfrak{g}_k$ has the AR property. Then G satisfies* Max-sn$_p^*$.

Lemma A was pointed out to me by K. A. Brown. The other fact about the property Max-sn$_p^*$ which we will need is contained in the next result.

LEMMA B. *Let p be a prime and $\mathfrak{X}$ a class of groups. Let G be a hyper-($\mathfrak{X}$ or*

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locally finite-p′) group such that G satisfies Max-sn$_p^*$. *Then there exists a finite chain* $1 = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$ *of normal subgroups* H_i *of G such that* H_i/H_{i-1} *is an* $\mathfrak{X}$*-group or a locally finite-p′ group for all* $1 \leqq i \leqq n$.

The problem of characterizing which locally nilpotent groups G are such that $\mathfrak{g}_k$ has the AR property for some field k is not hard given our earlier results and by using a theorem of Mal'cev we are able to give the following generalization of [7], theorem C′.

THEOREM C. *Let k be a field of characteristic* $p \geqq 0$ *and G a locally nilpotent group with maximal normal torsion p′-subgroup M. Then the following statements are equivalent.*

(i) $\mathfrak{g}_k$ *has the AR property.*

(ii) kG *satisfies the ascending chain condition on* $\mathfrak{g}_k$*-closed right ideals.*

(iii) G *satisfies* Max-sn$_p^*$.

(iv) G/M *is a nilpotent group each of whose upper central factors is an AR-p-group.*

The proof of Theorem C is so easy that we shall give it here. The implications (i) ⇒ (ii), (ii) ⇒ (iii) and (iv) ⇒ (i) are consequences of [7] corollary C2, the proof of Lemma A given below in §2, and [7], theorem C′, respectively. It remains to prove that (iii) ⇒ (iv). Suppose G satisfies Max-sn$_p^*$. By [4], vol. II, theorem 6.36, corollary 1, G/M is hypercentral and so a nilpotent group with the stated property (see the proof of [7], theorem C′).

Again with k a field of characteristic $p \geqq 0$, let G be a hyperabelian group. In view of [3], theorems 11.2.14 and 11.2.15, in considering when $\mathfrak{g}_k$ has the AR property one would expect to have to deal with the cases $p = 0$ and $p > 0$ separately. The case $p = 0$ is dealt with quite easily and we obtain:

THEOREM D. *Let k be a field of characteristic* 0 *and G a hyper-(Abelian or locally finite) group. Then* $\mathfrak{g}_k$ *has the AR property if and only if G contains a locally finite normal subgroup H such that G/H is a nilpotent group each of whose upper central factors has finite torsion-free rank.*

Now suppose $p > 0$. By a *p-chief factor* we shall mean a chief factor each of whose elements has finite order a power of p. It would seem reasonable in view of Lemma A and [5], corollary A3, to hope that, for G hyperabelian, $\mathfrak{g}_k$ has the AR property if and only if (a) G satisfies Max-sn$_p^*$ and (b) G centralizes all p-chief factors. We have been unable to prove this, but what we can prove is the following generalization of (7), theorem C′.

THEOREM E. *Let k be a field of characteristic $p > 0$ and G a hyper-(polycyclic or locally finite-p′) group. Then* $\mathfrak{g}_k$ *has the AR property if and only if G satisfies* Max-sn$_p^*$ *and G centralizes all p-chief factors.*

Before leaving [7], theorem C′, and its generalizations, let us note that Snider in [8], theorem B, has proved independently that if $R = kG$ where k is a field of characteristic $p \geqq 0$ and G is a nilpotent group each of whose upper central factors is AR-p-nilpotent, then $R_{\mathfrak{g}_k}$ is a (right and left) Noetherian ring.

For any prime p let $\mathrm{cch}_p(G)$ denote the intersection of the centralizers of the p-chief factors of G. Recall that if G is polycyclic then $\mathrm{cch}_p(G)$ has finite index in G. By generalizing this fact in our next result we can highlight Theorem E somewhat.

LEMMA F. *Let p be a prime and G a hyper-(Abelian or locally finite-p′) group such that G satisfies* Max-sn$_p^*$. *Then* $\mathrm{cch}_p(G)$ *has finite index in G.*

The remainder of the paper concerns the fAR property. The main result here is the following one. (Recall from [7] that an ideal I of a ring R has the fAR property if for every finitely generated right ideal E and finitely generated left ideal L there exists a positive integer n such that $E \cap I^n \leqq EI$ and $L \cap I^n \leqq IL$.)

THEOREM G. *Let k be a field and G an arbitrary group. Then* $\mathfrak{g} = \mathfrak{g}_k$ *has the fAR property if and only if*

$$\bigcap_{n=1}^{\infty} (E + \mathfrak{g}^n) = \{r \in kG : r(1-a) \in E \quad \text{for some } a \text{ in } \mathfrak{g}\}$$

for every finitely generated right ideal E of kG.

The relationship between the AR and fAR properties is as follows. For any field k and group G, necessary and sufficient conditions for $\mathfrak{g}_k$ to have the AR property are

(i) $\mathfrak{g}_k$ has the fAR property, and

(ii) kG satisfies the ascending chain condition on $\mathfrak{g}_k$-closed right ideals.

The necessity is given by [7], corollary C2. Conversely, if $\mathfrak{g}_k$ satisfies (i) and (ii), then by (ii) for any right ideal E of kG there exists a finitely generated right ideal F with $F \leqq E \leqq F^*$ (see the proof of [7], lemma 2.3). By (i) there exists a positive integer n such that $F \cap \mathfrak{g}^n \leqq F\mathfrak{g}$ and hence $E \cap \mathfrak{g}^n \leqq E\mathfrak{g}$. Here as we shall do elsewhere we write $\mathfrak{g}$ for $\mathfrak{g}_k$ when there is no ambiguity about the field k.

Finally we characterize for a given field k those Abelian groups G such that $\mathfrak{g}_k$ has the fAR property. Not surprisingly the characteristic of k again turns out to be crucial. For fields of characteristic zero we have:

THEOREM H. *Let k be a field of characteristic zero and G an arbitrary Abelian group. Then* $\mathfrak{g}_k$ *has the fAR property.*

The corresponding result for fields of non-zero characteristic is the following one.

THEOREM I. *Let k be a field of characteristic* $p > 0$ *and G an Abelian group. Then the following statements are equivalent.*

(i) $\mathfrak{g}_k$ *has the fAR property.*

(ii) *For every finitely generated subgroup N of G the group G/N has no p-elements of infinite p-height.*

(iii) *For every finitely generated subgroup N of G there exists a positive integer n such that* $N \cap G^{p^n} \leqq N^p$.

By a *p-element* we mean an element with finite order a power of p. As usual, G^{p^n} is the subgroup $\{x^{p^n}: x \in G\}$ of G and an element y has *infinite p-height* if

$$y \in \bigcap_{n=1}^{\infty} G^{p^n}.$$

Combining Theorems H and I we see that if G is a free Abelian group then $\mathfrak{g}_k$ has the fAR property for any field k.

We use [4] as a general reference for group theoretic terminology.

2. Proofs of Lemmas A, B and F and Theorem D

Lemmas A, B and F and Theorem D are quite easy to prove and so we deal with them all in this section.

PROOF OF LEMMA A. Suppose k is a field of characteristic $p \geqq 0$ and G a group such that $\mathfrak{g} = \mathfrak{g}_k$ has the AR property. Let $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ be an ascending chain of subgroups of G such that H_i is normal in H_{i+1} for all $i \geqq 1$. Let $H = \bigcup_{i=1}^{\infty} H_i$. In the notation of [7] there exists a finitely generated right ideal E of kG such that $E \leqq \bar{\mathfrak{h}} \leqq E^*$ (see [7], corollary C2), where $\bar{\mathfrak{h}} = \mathfrak{h}(kG)$. Clearly there exists a positive integer n_0 such that $E \leqq \bar{\mathfrak{h}}_{n_0}$. Let $n \geqq n_0$. Let S be a finite subset of H_{n+1}. For each element s of S there exists a in $\mathfrak{g}$ such that $(1-s)(1-a) \in \bar{\mathfrak{h}}_n$. But by [7], lemma 2.2, kG satisfies the right Ore condition with respect to $\{1-b: b \in \mathfrak{g}\}$, and so there exists an element c in $\mathfrak{g}$ such that $(1-s)(1-c) \in \bar{\mathfrak{h}}_n$ for all s in S. Let T be a transversal to the right cosets of H_{n+1} in G. Then $kG = \bigoplus_{t \in T} (kH_{n+1})t$ and $\bar{\mathfrak{h}}_n = \bigoplus_{t \in T} \mathfrak{h}_n(kH_{n+1})t$. It follows that there exists d in $\mathfrak{h}_{n+1}$ such that $(1-s)(1-d) \in \mathfrak{h}_n(kH_{n+1})$ for all s in S. Let U be the subgroup of H_{n+1} generated by S and H_n. Then the augmentation ideal $\mathfrak{u}$ of kU satisfies

$$\mathfrak{u} = \sum_{s \in S} kU(1-s) + (kU)\mathfrak{h}_n.$$

Thus $\mathfrak{u}(1-d) \leqq (kU)\mathfrak{h}_n$ and it follows, by the proof of [7], theorem B, that U/H_n is a finite-p' group. Hence H_{n+1}/H_n is a locally finite-p' group for all $n \geqq n_0$.

Proof of Lemma B. Every non-trivial homomorphic image of G contains a non-trivial normal subgroup which is either an $\mathfrak{X}$-group or a locally finite-p' group. If G contains a non-trivial locally finite-p' normal subgroup let H_1 be the maximal locally finite-p' normal subgroup of G (see [4], vol. I, theorem 1.45, corollary). If $H_1 \neq G$ then G/H_1 does not contain any non-trivial locally finite-p' normal subgroups. Let H_2 be a normal subgroup of G containing H_1 such that H_2/H_1 is a non-trivial $\mathfrak{X}$-group. If $H_2 \neq G$ define H_3 to be a normal subgroup properly containing H_2 such that H_3/H_2 is the maximal locally finite-p' subgroup of G/H_2 if this is non-trivial and an $\mathfrak{X}$-group otherwise. If $H_3 \neq G$ then repeat the process. Note that if in the ascending chain $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ each of the groups H_2/H_1 and H_3/H_2 is an $\mathfrak{X}$-group there will be an integer $n \geqq 3$ such that H_{n+1}/H_n is a locally finite-p' group because G satisfies Max-sn$_p^*$. In this way we obtain an ascending chain of normal subgroups $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ of normal subgroups such that for each $i \geqq 1$ H_{i+1}/H_i is an $\mathfrak{X}$-group or the maximal locally finite-p' normal subgroup of G/H_i. Since G satisfies Max-sn$_p^*$ this process stops and $H_n = G$ for some positive integer n. Now suppose that G does not contain a non-trivial locally finite-p' normal subgroup. In this case let $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ be an ascending chain of normal subgroups such that H_{i+1}/H_i is a non-trivial $\mathfrak{X}$-group for each $i \geqq 1$ until we reach a normal subgroup H_j such that G/H_j contains a non-trivial locally finite-p' normal subgroup and then define H_{j+1} to be the normal subgroup containing H_j such that H_{j+1}/H_j is the maximal locally finite-p' normal subgroup of G/H_j. And then proceed as before to produce the required chain of normal subgroups. This proves Lemma B.

Proof of Theorem D. *Let* k be a field of characteristic 0 and G a group. Suppose first that G has a locally finite normal subgroup H such that G/H is nilpotent with each of its upper central factors having finite torsion-free rank. If $Q = G/H$ then $\mathfrak{q}$ has the AR property ([7], theorem C′). If E is a right ideal of kG then there exists a positive integer n such that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g} + \bar{\mathfrak{h}}$. But if $r \in \bar{\mathfrak{h}}$

then there exists a in $\mathfrak{g}$ with $r(1-a)=0$. It follows that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g}$ and $\mathfrak{g}$ has the AR property.

Conversely, suppose that $\mathfrak{g}$ has the AR property. By Lemmas A and B, G has a finite series.

$$(1) \qquad 1 = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$$

of normal subgroups H_i such that H_i/H_{i-1} is Abelian or locally finite for all $1 \leqq i \leqq n$. Moreover, since G satisfies Max-sn$_0^*$ by Lemma A, if we suppose merely that H_{i-1} is normal in H_i then we can suppose without loss of generality that H_i/H_{i-1} is infinite cyclic or locally finite for each $1 \leqq i \leqq n$.

Let D_m denote the mth dimension subgroup of G, that is

$$D_m = G \cap (1 + \mathfrak{g}^m).$$

Then $G = D_1 \geqq D_2 \geqq \cdots$ is a central series for G and D_i/D_{i+1} is torsion-free or trivial for each $i \geqq 1$ (see [3], lemma 3.3.2). Consider the series

$$(2) \qquad 1 \leqq D_{n+2} \leqq D_{n+1} \leqq \cdots \leqq D_2 \leqq D_1 = G.$$

By [6], 2.10.1, the series (1) and (2) have equivalent refinements and it follows that D_{n+1}/D_{n+2} is not torsion-free (see the proof of [6], 7.1.5). Thus $D_{n+1} = D_{n+2} = \cdots$ and if $T = D_{n+1}$ then

$$\mathfrak{t} \leqq \bigcap_{s=1}^{\infty} \mathfrak{g}^s.$$

Thus, by the proof of Lemma A, T is a locally finite group. If $Q = G/T$ then Q is nilpotent and $\mathfrak{q}$ has the AR property, so the result follows by [7], theorem C′.

Proof of Lemma F. By Lemma B there exists a finite chain (1) such that H_i is a normal subgroup of G and H_i/H_{i-1} is Abelian or locally finite-p' for all $1 \leqq i \leqq n$. Let $H = H_1$. By induction on n we can suppose that there exists a normal subgroup C_1 of finite index in G such that $H \leqq C_1$ and C_1 centralizes all p-chief factors of G/H. If H is a locally finite-p' group then C_1 centralizes all p-chief factors of G. So suppose H is Abelian. Then H/H^p is a finite group because G satisfies Max-sn$_p^*$. Let C_2 be the centralizer in G of H/H^p and let $C = C_1 \cap C_2$. Then C is a normal subgroup of finite index in G. Let $h \in H$, $x \in C$. Then

$$[h^p, x] = [h^{p-1}, x]^h [h, x] = [h^{p-1}, x][h, x] = [h, x]^p \in H^{p^2}$$

since H is Abelian. It follows that C centralizes H^p/H^{p^2} and by the same argument C centralizes

$$H^{p^m}/H^{p^{m+1}}$$

for all positive integers m. By [7], theorem C, $\mathfrak{h}$ has the AR property in kH and it follows that, for any homomorphic image $\bar{H}$ of H,

$$\bigcap_{m=1}^{\infty} \bar{H}^{p^m}$$

is a locally finite-p' group (to see this apply [7], lemma 2.1 and use the argument at the end of the proof of Lemma A). Thus C centralizes all p-chief factors of G.

3. Proof of Theorem E

In contrast to the results proved in §2 the proof of Theorem E is rather long and involved. We begin this section by stating and proving some preliminary lemmas.

Let R be a ring and I an ideal of R. It is well known that I has the AR property if and only if given any essential submodule N of a finitely generated R-module M such that $NI = 0$, there exists a positive integer n such that $MI^n = 0$ (see [2], 2.7 and 2.8, and note that R need not be Noetherian). As a result of this fact we have the following known result.

Lemma 3.1. *If an ideal I of a ring R has the AR property then for every submodule N of a finitely generated R-module M there exists a positive integer n such that $N \cap MI^n \leqq NI$.*

The above characterization of the AR property has a second consequence. Recall that an ideal I of a ring R has a *centralizing set of generators* if there exists a finite chain

$$0 = I_0 \leqq I_1 \leqq \cdots \leqq I_n = I$$

of ideals I_j of R such that I_j/I_{j-1} is generated by a finite collection of central elements of the ring R/I_{j-1} for all $1 \leqq j \leqq n$. By adapting the proof of [2], 2.7, we have:

Lemma 3.2. *Let R be a Noetherian ring and $I \geqq J$ ideals of R such that I/J has the AR property in the ring R/J and J has a centralizing set of generators. Then I has the AR property.*

Now we return to group rings and prove a lemma which really goes back to [5].

LEMMA 3.3. *Let k be a field of characteristic $p > 0$ and G a group such that $\mathfrak{g}_k$ has the AR property. Then G centralizes all p-chief factors.*

PROOF. It is sufficient to prove that if X is a minimal normal subgroup of G such that every element of X is a p-element then $[X, G] = 1$. Suppose $[X, G] \neq 1$. Then $X = [X, G]$ and it follows that

$$X \leqq \bigcap_{n=1}^{\infty} G_n$$

where $G = G_1 \geqq G_2 \geqq \cdots$ is the lower central series of G. But it is well known that $\mathfrak{g}_n \leqq \mathfrak{g}^n$ for all $n \geqq 1$ and hence

$$\mathfrak{x} \leqq \bigcap_{n=1}^{\infty} \mathfrak{g}^n.$$

By [7], lemma 2.1 and the proof of corollary 3.2, it follows that $X = 1$, a contradiction.

LEMMA 3.4. *Let k be a field and G a group such that $\mathfrak{g}_k$ has the AR property (fAR property) and let H be a subgroup of G. Then $\mathfrak{h}_k$ has the AR property (fAR property).*

PROOF. We prove the result for the case of the AR property, the proof for the other case being similar. Let T be a transversal to the right cosets of H in G. Let E be a right ideal of the ring kH and $\bar{E} = E(kG)$. Then $\bar{E} = \bigoplus_{t \in T} Et$. By hypothesis there exists a positive integer n such that $\bar{E} \cap \mathfrak{g}^n \leqq \bar{E}\mathfrak{g}$. Let $e \in E \cap \mathfrak{h}^n$. Then there exist a positive integer m and elements e_i of E and x_i of G $(1 \leqq i \leqq m)$ such that

$$e = \sum_{i=1}^{m} e_i(1 - x_i).$$

For each $1 \leqq i \leqq m$ there exist elements h_i of H and t_i of T such that $x_i = h_i t_i$. There are two cases to consider. First suppose that none of the elements x_i belongs to H. Then

$$e = \sum_{i=1}^{m} e_i - \sum_{i=1}^{m} e_i h_i t_i$$

implies that

$$\sum_{i=1}^{m} e_i h_i = 0$$

and so

$$e = \sum_{i=1}^{m} e_i(1 - h_i).$$

Now suppose that there exists an integer s with $1 \leqq s \leqq m$ such that $x_i \in H$ $(1 \leqq i \leqq s)$ but $x_i \notin H$ $(s+1 \leqq i \leqq m)$. Then without loss of generality we can take $t_i = 1$ $(1 \leqq i \leqq s)$. Hence

$$e = \sum_{i=1}^{s} e_i(1 - h_i) + \sum_{i=s+1}^{m} e_i - \sum_{i=s+1}^{m} e_i h_i t_i.$$

It follows that

$$\sum_{i=s+1}^{m} e_i h_i = 0$$

and again we have

$$e = \sum_{i=1}^{m} e_i(1 - h_i).$$

Thus $e \in E\mathfrak{h}$ and it follows that

$$E \cap \mathfrak{h}^n \leqq E\mathfrak{h}.$$

Thus $\mathfrak{h}$ has the AR property.

The next group theoretic lemma turns out to be crucial.

LEMMA 3.5. Let G be a poly-(polycyclic or locally finite) group of automorphisms of a polycyclic group. Then G is polycyclic-by-finite.

PROOF.† Let $1 = G_0 \leqq G_1 \leqq \cdots \leqq G_n = G$ be a chain of subgroups of G such that G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is polycyclic or locally finite $(1 \leqq i \leqq n)$. Let $H = G_{n-1}$. By induction on n it is sufficient to prove that if H is polycyclic-by-finite and G/H is locally finite then G is polycyclic-by-finite (see

† The referee has pointed out that the proof of Lemma 3.5 does not require the Hall–Kulatilaha–Kargapolov Theorem, but the result can be proved using the fact that the automorphism group of a polycyclic group is a subgroup of $GL_n(\mathbf{Z})$ (see In. I. Merzltakov, *Integer representations of the holomorphs of polycyclic groups*, Algebra i Logika 9 (1970), 539–558 (Russian); also B. A. F. Wehrfritz, *On the holomorphs of soluble groups of finite rank*, J. Pure Appl. Algebra 4 (1974), 55–69). For it suffices to assume that G is polycyclic-by-locally finite. If N is the maximal soluble normal subgroup of G, then N is closed and hence G/N is a subgroup of $GL_n(Q)$. But periodic subgroups of $GL_n(Q)$ are finite (B. A. F. Wehrfritz, *Infinite Linear Groups* (Springer–Verlag), 9.33). Thus G/N is finite. Also N is polycyclic since it is a soluble subgroup of $GL_n(\mathbf{Z})$.

[6], 7.1.10). By [6], 7.1.7, there is a characteristic subgroup N of H such that N is polycyclic and H/N is finite. Then N is a normal subgroup of G. Moreover, the group G/N is locally finite by [4], vol. I, theorem 1.45. If G/N is finite then the result is proved. So suppose that G/N is infinite. Then by the Hall–Kulatilaka–Kargapolov Theorem ([4], vol. I, theorem 3.43) there exists a subgroup X of G containing N such that X/N is infinite Abelian. Then X is a soluble group and by [4], vol. I, theorem 3.27, X is polycyclic. But this implies that X/N is finitely generated and hence finite, a contradiction.

LEMMA 3.6. *Let p be a prime and G a hyper-(Abelian or locally finite-p') group such that G satisfies* Max-sn$_p^*$ *and let H be a normal subgroup of G. If $G = \mathrm{cch}_p(G)$ then $H = \mathrm{cch}_p(H)$.*

PROOF. By Lemma B there exists a finite chain

$$1 = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$$

of normal subgroups H_i of G such that H_i/H_{i-1} is an Abelian group with no non-trivial p'-elements or a locally finite-p' group for all $1 \leqq i \leqq n$. We prove the result by induction on n. Let $N = H_1$. By induction HN/N, and hence $H/(H \cap N)$, centralizes all p-chief factors. If N is a locally finite-p' group then $H = \mathrm{cch}_p(H)$. So suppose that N is Abelian. By [7], theorem C, $\mathfrak{n}$ has the AR property in kN and so

$$\bigcap_{m=1}^{\infty} N^{p^m} = 1.$$

Since $G = \mathrm{cch}_p(G)$ and $N^{p^m}/N^{p^{m+1}}$ is finite it follows that G acts nilpotently on $N^{p^m}/N^{p^{m+1}}$ for all integers $m \geqq 1$. Hence $H = \mathrm{cch}_p(H)$.

Before proceeding to the proof of Theorem E there are two technical matters to dispose of. The first concerns quotient rings. Let k be a field and H a normal subgroup of a group G. Let $R = kG$, $S = kH$. If $\mathfrak{h}$ has the AR property then S satisfies the right and left Ore conditions with respect to $T = \{1 - a : a \in \mathfrak{h}\}$. If

$$I = \{s \in S : st = 0 \text{ for some } t \text{ in } T\}$$

then I is an ideal of S,

$$I = \{s \in S : ts = 0 \text{ for some } t \text{ in } T\},$$

and the element $t + I$ of S/I is regular for all t in T (see [7], §1). The partial quotient ring of S with respect to T is denoted by $S_{\mathfrak{h}}$. Now, because T is G-invariant, R satisfies the right and left Ore conditions with respect to T (see the proof of [3], lemma 13.3.5 (ii)). Let

$$J = \{r \in R : rt = 0 \text{ for some } t \text{ in } T\}.$$

Then J is an ideal of R and

$$J = IR = RI = \{r \in R : tr = 0 \text{ for some } t \text{ in } T\}.$$

Thus the elements $t + J$ $(t \in T)$ of R/J are all regular and we can form the partial quotient ring of R with respect to T which we shall denote by $R_\flat$. If A is an ideal of R then we write $AR_\flat$ for the right ideal $((A+J)/J)R_\flat$ of $R_\flat$.

The other matter concerns the strong AR property. Let R be a right Noetherian ring and I an ideal of R. Then I has the (*right*) *strong AR property* if for each finitely generated right R-module M and submodule N there exists a positive integer m such that

$$N \cap MI^n = (N \cap MI^m)I^{n-m}$$

for all integers $n \geqq m$. Note that by taking $n = m + 1$ we have in particular that $N \cap MI^{m+1} \leqq NI$. Let $R(I)$ denote the subring of the polynomial ring $R[X]$ given by

$$R(I) = R + IX + I^2X^2 + \cdots.$$

Recall that if the ideal I of the right Noetherian ring R is generated by central elements then $R(I)$ is right Noetherian. Recall further that for any ideal I, $R(I)$ is right Noetherian implies that I has the strong AR property (see [3], lemma 11.2.1).

PROOF OF THEOREM E. Let k be a field of characteristic $p > 0$ and G a group such that $\mathfrak{g}$ has the AR property. Then G satisfies Max-sn$_p^*$ by Lemma A and G centralizes all p-chief factors by Lemma 3.3.

Conversely, suppose that G is a hyper-(polycyclic or locally finite-p') group such that G satisfies Max-sn$_p^*$ and G centralizes all p-chief factors. By Lemma B there exists a chain

$$1 = H_0 \leqq H_1 \leqq \cdots \leqq H_n = G$$

of normal subgroup H_i of G such that H_i/H_{i-1} is finitely generated Abelian or locally finite-p' for all $1 \leqq i \leqq n$. We prove the result by induction on n. The case $n = 1$ is trivial. So suppose $n > 1$ and let $H = H_{n-1}$ and $N = H_1$. If $Q = G/N$ then $\mathfrak{q}$ has the AR property. Suppose that N is a locally finite-p' group. If E is a right ideal of $R = kG$ then there exists a positive integer m such that

$E \cap \mathfrak{g}^m \leqq E\mathfrak{g} + \bar{\mathfrak{n}}$. But if $r \in \bar{\mathfrak{n}}$ then there exists a in $\mathfrak{g}$ such that $r(1-a) = 0$. It follows that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g}$ and hence $\mathfrak{g}$ has the AR property. Thus we can suppose that N is finitely generated Abelian.

Now suppose that N is contained in the centre of G. By Lemma 3.6 $\mathfrak{h}$ has the AR property. We next show that if G/H is locally finite-p' then $\mathfrak{g}$ has the AR property. As a first step we shall show that $\mathfrak{g}$ has the fAR property. Let E be a finitely generated right ideal of R. Let X be the subgroup of G generated by H and the supports of the elements in a finite generating set for E and let $S = kX$. Then X/H is a finite group and by Lemma 3.1 there exists a positive integer m such that $(E \cap S) \cap S\mathfrak{h}^m \leqq (E \cap S)\mathfrak{h}$. If T is a transversal to the right cosets of X in G then $E = \bigoplus_{t \in T}(E \cap S)t$ and $\bar{\mathfrak{h}}^m = \bigoplus_{t \in T} S\mathfrak{h}^m t$ and it follows that $E \cap \bar{\mathfrak{h}}^m \leqq E\bar{\mathfrak{h}}$. Let

$$A = H^{p^m}.$$

Then $\mathfrak{a} \leqq \mathfrak{h}^m$ and H/A is a finite-p group. If B is a subgroup of G containing H such that B/H is finitely generated then B/H is a finite-p' group. We claim that B/A is p-nilpotent, that is B/A is an extension of a p'-group by a p-group. The proof is by induction on the order of B/A. Let Z be a normal subgroup of G such that $A < Z \leqq H$ and Z/A is a central p-subgroup of G/A. By induction there exists a normal subgroup C of B such that $Z \leqq C$, C/Z is a p'-group and B/C a p-group. By [4], vol. I, theorem 4.12, C/A is p-nilpotent. Hence B/A is p-nilpotent. Let $L = B/A$. Then kL satisfies the right Ore condition with respect to $\{1-c: c \in \mathfrak{l}\}$ ([3], theorem 11.2.15). Hence $k(G/A)$ satisfies the right Ore condition with respect to $\{1-p: p \in \mathfrak{p}\}$, where $P = G/A$. By the proof of [7], lemma 7.3 it follows that for each element r of $\mathfrak{g}^m$ there exists a in $\mathfrak{g}$ such that $r(1-a) \in \bar{\mathfrak{h}}^m + \bar{\mathfrak{a}} \leqq \bar{\mathfrak{h}}^m$. Hence $E \cap \mathfrak{g}^m \leqq E\mathfrak{g}$. It follows that $\mathfrak{g}$ has the fAR property.

It is quite easy to prove now that $\mathfrak{g}$ has the AR property. For, $\mathfrak{q}$ has the AR property, where $Q = G/N$, and N is contained in the centre of G. By induction on the number of generators of N it follows that $\mathfrak{g}$ has the AR property (see the proof of [7], lemma 7.3).

Still assuming that N is contained in the centre of G, now suppose that G/H is finitely generated Abelian. Let $U = kH$. Since $\mathfrak{h}$ has the AR property it follows that the ring $U_{\mathfrak{h}}$ is Noetherian (see [7], corollary C1). By our remarks before this proof we know that we can form the ring $R_{\mathfrak{h}}$. Since G/H is finitely generated it follows by [3], theorem 10.2.6, that $R_{\mathfrak{h}}$ is Noetherian. The ideal $\mathfrak{n}R_{\mathfrak{h}}$ is generated by central elements and the ideal $\mathfrak{g}R_{\mathfrak{h}}/\mathfrak{n}R_{\mathfrak{h}}$ has the AR property. By Lemma 3.2 the ideal $\mathfrak{g}R_{\mathfrak{h}}$ has the AR property and hence $\mathfrak{g}$ has the AR property.

Thus if N is contained in the centre of G then $\mathfrak{g}$ has the AR property. Now suppose that N is not necessarily contained in the centre of G. Let C be the centralizer of N in G. Then $N \leqq C$. By Lemma 3.4 the augmentation ideal of $k(C/N)$ has the AR property and by the above argument $\mathfrak{c}$ has the AR property. Let J be the ring $(kC)_{\mathfrak{c}}$. By [7], corollary C1, J is a Noetherian ring. Moreover, by Lemma 3.5 the group G/C is polycyclic-by-finite. Since N is contained in the centre of C it follows that $J(\mathfrak{n}J)$ is Noetherian ([3], lemma 11.2.1) and, because G/C is polycyclic-by-finite, the ring $R_{\mathfrak{c}}(\mathfrak{n}R_{\mathfrak{c}})$ is Noetherian ([3], theorem 10.2.6). Hence the ideal $\mathfrak{n}R_{\mathfrak{c}}$ has the strong AR property in the ring $R_{\mathfrak{c}}$ and it follows that if E is a right ideal of R then there exists a positive integer m such that $E \cap \mathfrak{n}^m \leqq E\mathfrak{c} \leqq E\mathfrak{g}$. Let

$$Z = N^{p^m}.$$

Then $\mathfrak{z} \leqq \mathfrak{n}^m$ and the group N/Z is a finite p-group. But G centralizes p-chief factors and so by induction on the order of N/Z we can suppose that N/Z is contained in the centre of G/Z. By the above argument it follows that the augmentation ideal of $k(G/Z)$ has the AR property. Thus there exists a positive integer s such that $E \cap \mathfrak{g}^s \leqq E\mathfrak{g} + \mathfrak{z}$ from which we deduce that $E \cap \mathfrak{g}^s \leqq E\mathfrak{g} + E \cap \bar{\mathfrak{n}}^m \leqq E\mathfrak{g}$. This completes the proof of Theorem E.

4. Proof of Theorem G

Let k be a field, G a group and R the group algebra kG. For each right ideal F and element r of R define

$$h_F(r)$$

to be the greatest positive integer n such that $r \in F + \mathfrak{g}^n$ if such an n exists and ∞ otherwise, where we make the usual convention that $\mathfrak{g}^0 = R$.

Let E be a finitely generated right ideal of R and suppose that E can be generated by n but no fewer elements. Choose a_1 in E such that $h_0(a_1)$ is minimal in

$$\{h_0(r)\colon r \text{ belongs to an } n\text{-generating set of } E\}.$$

By an *n-generating set* of E we mean a set of n elements which generated E as a right ideal. Let $A_0 = 0$ and $A_1 = a_1R$. Choose a_2 in E such that $h_{A_1}(a_2)$ is minimal in

$$\{h_{A_1}(r)\colon a_1, r \text{ belong to the same } n\text{-generating set of } E\}.$$

Repeating this process, define a sequence of elements $a_1, a_2, \cdots$ of E and right ideals $A_i = a_1R + \cdots + a_iR$ $(1 \leqq i \leqq n)$ such that for each $i \geqq 1$, $h_{A_i}(a_{i+1})$ is minimal in

$$\{h_{A_i}(r)\colon a_1, \cdots, a_i, r \text{ belong to the same } n\text{-generating set of } E\}.$$

In this way we produce an ordered collection of elements $a_1, \cdots, a_n$ of E such that $E = a_1R + \cdots + a_nR$. We call such a set of elements a *minimal generating set* of E. The crucial property of these sets is given in the next result.

LEMMA 4.1. *In the above notation* $h_{A_0}(a_1) \leqq h_{A_1}(a_2) \leqq \cdots \leqq h_{A_{n-1}}(a_n)$.

PROOF. Suppose on the contrary that there exist $0 \leqq i < j \leqq n-1$ such that

$$h_{A_j}(a_{j+1}) < h_{A_i}(a_{i+1}).$$

Let $h_{A_j}(a_{j+1}) = s$ and $h_{A_i}(a_{i+1}) = t$. Then there exist elements r_u $(1 \leqq u \leqq j)$ of R and b of $\mathfrak{g}^s$ such that

$$a_{j+1} = a_1r_1 + \cdots + a_jr_j + b.$$

Let

$$c = a_{j+1} - a_{i+1}r_{i+1} - \cdots - a_jr_j.$$

Then

$$h_{A_i}(c) = s < t$$

and $a_1, \cdots, a_i, c$ belongs to an n-generating set of E. This contradicts the choice of a_{i+1}. This proves the lemma.

Our next claim is that without loss of generality we may suppose that if $h_{A_i}(a_{i+1})$ is finite then

$$h_{A_0}(a_{i+1}) = h_{A_i}(a_{i+1}). \tag{3}$$

For, let $0 \leqq i \leqq n-1$ and suppose that $h_{A_i}(a_{i+1}) = m$. Then $a_{i+1} \in A_i + \mathfrak{g}^m$ but $a_{i+1} \notin A_i + \mathfrak{g}^{m+1}$. There exists d in A_i such that $a_{i+1} - d \in \mathfrak{g}^m$. Let $a'_{i+1} = a_{i+1} - d$. Then

$$h_{A_0}(a'_{i+1}) = h_{A_i}(a'_{i+1}) = m,$$

and we can replace a_{i+1} by a'_{i+1}. From now on we shall assume that any minimal generating set $\{a_1, \cdots, a_n\}$ of E has the additional property (3).

Let $S = \{a_1, \cdots, a_n\}$ be a minimal generating set for a right ideal E. Define

$$\nu(S) = 1 + h_{A_i}(a_{i+1})$$

where i is the greatest integer such that $h_{A_i}(a_{i+1}) < \infty$, and, if no such i exists, define $\nu(S) = 1$. Define $\nu(E)$ to be the minimal value of $\nu(S)$ as S runs through the minimal generating sets of E.

We shall say that g has the *finite intersection property* if

$$\bigcap_{s=1}^{\infty} (E + \mathfrak{g}^s) = \{r \in R \colon r(1-a) \in E \text{ for some } a \text{ in } \mathfrak{g}\}$$

for every finitely generated right ideal E of R. Then Theorem G follows by [7], lemma 2.1, and the next result.

LEMMA 4.2. *In the above notation if $\mathfrak{g}_k$ has the finite intersection property then for any finitely generated right ideal E, $E \cap \mathfrak{g}^{\nu(E)} \leqq E\mathfrak{g}$.*

PROOF. Suppose $\nu(E) = 1$ and let $S = \{a_1, \cdots, a_n\}$ be a minimal generating set of E with $\nu(S) = 1$. Then either $a_1 \notin \mathfrak{g}$ or $S \leqq \bigcap_{n=1}^{\infty} \mathfrak{g}^n$. If $S \leqq \bigcap_{n=1}^{\infty} \mathfrak{g}^n$ then $E \leqq \bigcap_{n=1}^{\infty} \mathfrak{g}^n$. If $e \in E\cap\mathfrak{g}$ it follows that there exists a in g such that $e(1-a) = 0$ and hence $e \in E\mathfrak{g}$. Thus $E \cap \mathfrak{g} \leqq E\mathfrak{g}$. So suppose that $a_1 \notin \mathfrak{g}$. Then there exists q in k such that $qa_1 \in 1 + \mathfrak{g}$ and so without loss of generality we can suppose that $a_1 = 1 - u$ for some element u of $\mathfrak{g}$. Since kG satisfies the right Ore condition with respect to $\{1 - v \colon v \in \mathfrak{g}\}$ (see [7], lemma 2.2) it follows that for each element e of $E \cap \mathfrak{g}$ there exists v in $\mathfrak{g}$ such that $e(1-v) = a_1 r$ for some r in kG. But $a_1 r \in \mathfrak{g}$ implies $r \in \mathfrak{g}$ and it follows that $e \in E\mathfrak{g}$. Thus $E \cap \mathfrak{g} \leqq E\mathfrak{g}$.

Now suppose that $\nu(E) > 1$ and $S = \{a_1, \cdots, a_n\}$ is a minimal generating set with $\nu(S) = \nu(E)$. Choose (if possible) n_0 such that $1 \leqq n_0 < n$ and

$$h_{A_i}(a_{i+1}) = \infty$$

for all $n_0 \leqq i \leqq n-1$. By induction on n, if $F = a_1 R + \cdots + a_{n-1} R$ then

$$F \cap \mathfrak{g}^{\nu(F)} \leqq F\mathfrak{g}.$$

But $a_n \in \bigcap_{n=1}^{\infty} (F + \mathfrak{g}^n)$ implies that $E \leqq F^*$ and hence $E \cap \mathfrak{g}^{\nu(F)} \leqq E\mathfrak{g}$, and the result is proved in this case since $\nu(E) = \nu(F)$.

Finally consider the case when $h_{A_{n-1}}(a_n) < \infty$. Let $F = a_1 R + \cdots + a_{n-1} R$. By induction on n, $F \cap \mathfrak{g}^{\nu(F)} \leqq F\mathfrak{g}$. Let $e \in E \cap \mathfrak{g}^{\nu(E)}$. Then $e = f + a_n r$ for some f in F and r in R. Thus $a_n r \in F + \mathfrak{g}^{\nu(E)}$. If $r \notin \mathfrak{g}$ then there exists r' in R such that $rr' \in 1 + \mathfrak{g}$ and hence $a_n \in F + \mathfrak{g}^{\nu(E)}$, a contradiction. Thus $r \in \mathfrak{g}$ and $f = e - a_n r \in F \cap \mathfrak{g}^{\nu(E)} \leqq F\mathfrak{g}$ by Lemma 4.1 and (3). It follows that $e \in E\mathfrak{g}$ and $E \cap \mathfrak{g}^{\nu(E)} \leqq E\mathfrak{g}$, as required.

5. Proof of Theorems H and I

To prove Theorems H and I we require three lemmas.

LEMMA 5.1. *Let k be a field of characteristic $p \geqq 0$ and G an Abelian group with a subgroup H such that G/H is a torsion p'-group. Then $\mathfrak{g}_k$ has the fAR property (in kG) if and only if $\mathfrak{h}_k$ has the fAR property (in kH).*

PROOF. If $\mathfrak{g}$ has the fAR property then so has $\mathfrak{h}$ by Lemma 3.4. Conversely, suppose that $\mathfrak{h}$ has the fAR property. To prove that $\mathfrak{g}$ has the fAR property suppose first that G/H is a finite group. Suppose that G/H has order n. Define $\varphi: G \to H$ by $\varphi(x) = x^n$ $(x \in G)$. Then φ is a homomorphism and the kernel N of φ is a p'-subgroup of G. If $Q = G/N$ then Q is isomorphic to a subgroup of H and by Lemma 3.4 it follows that $\mathfrak{q}$ has the fAR property. If E is a finitely generated ideal of kG then there exists a positive integer m such that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g} + \bar{\mathfrak{n}}$. Since N is a p'-group it follows that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g}$ (see the proof of Theorem E).

Now suppose that G/H is a torsion p'-group. Let E be a finitely generated ideal of kG. Let T be the subgroup of G generated by H and the supports of the finite collection of elements in a finite generating set for E. Then T/H is finite and by the above argument $\mathfrak{t}$ has the fAR property. Since $E \cap kT$ is a finitely generated ideal of kT it follows that there exists a positive integer m such that $(E \cap kT) \cap \mathfrak{t}^m \leqq (E \cap kT)\mathfrak{t}$. If U is a transversal to the cosets of T in G then $E = \bigoplus_{u \in U}(E \cap kT)u$ and $\bar{\mathfrak{t}}^m = \bigoplus_{u \in U}\mathfrak{t}^m u$. Hence $E \cap \bar{\mathfrak{t}}^m \leqq E\mathfrak{t}$. Now let $e \in E \cap \mathfrak{g}^m$. For each element r of $\mathfrak{g}$ there exists a in $\mathfrak{g}$ such that $r(1-a) \in \bar{\mathfrak{t}}$. Hence there exists b in $\mathfrak{g}$ with $e(1-b) \in E \cap \bar{\mathfrak{t}}^m \leqq E\bar{\mathfrak{t}}$. This implies that $e \in E\mathfrak{g}$. It follows that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g}$ and hence $\mathfrak{g}$ has the fAR property.

LEMMA 5.2. *Let J be a ring and H and N normal subgroups of a group G such that $H \cap N = 1$. If E is a right ideal of JG such that $E = (E \cap JH)JG$ then $E \cap \bar{\mathfrak{n}}_J \leqq E\mathfrak{n}_J$.*

PROOF. Let T be a transversal to the right cosets of the normal subgroup $H \times N$ in G. Let $e \in E \cap \bar{\mathfrak{n}}_J$. Then there exist a positive integer m and elements e_i of $E \cap JH$ and r_i of JG $(1 \leqq i \leqq m)$ such that

$$e = \sum_{i=1}^{m} e_i r_i.$$

For each $1 \leqq i \leqq m$, r_i is a finite sum

$$\sum_j s_{ij} x_{ij} t_{ij}$$

where $s_{ij} \in JN$, $x_{ij} \in H$ and $t_{ij} \in T$. Let δ: $JN \to J$ be the canonical homomorphism. Then

$$e = \sum_i e_i \sum_j \{s_{ij} - \delta(s_{ij})\}x_{ij}t_{ij} + \sum_{i,j} e_i\delta(s_{ij})x_{ij}t_{ij}.$$

If

$$u = \sum_{i,j} e_i\delta(s_{ij})x_{ij}t_{ij}$$

then, because $s_{ij} - \delta(s_{ij}) \in \mathfrak{n}$, we have $u \in \mathfrak{n}$ and hence $u = 0$. It follows that $e \in E\mathfrak{n}$, as required.

LEMMA 5.3. *Let k be a field and G an Abelian group such that every finitely generated subgroup is contained in a finitely generated direct factor. Then $\mathfrak{g}_k$ has the fAR property.*

PROOF. Let E be a finitely generated ideal of kG. Let X be the finitely generated subgroup of G which is generated by the supports of a finite set of generators of E. By hypothesis there exists a finitely generated subgroup H and a subgroup N such that $X \leqq H$ and $G = H \times N$. If $Q = G/N$ then Q is a finitely generated Abelian group and hence $\mathfrak{q}$ has the AR property. It follows that there exists a positive integer m such that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g} + \bar{\mathfrak{n}}$. But $E \cap \bar{\mathfrak{n}} \leqq E\mathfrak{n} \leqq E\mathfrak{g}$ by Lemma 5.2 and it follows that $E \cap \mathfrak{g}^m \leqq E\mathfrak{g}$. Thus $\mathfrak{g}$ has the fAR property.

By combining Lemmas 5.1 and 5.3 we see at once that if k is a field of characteristic 0 and G an arbitrary group then $\mathfrak{g}_k$ has the fAR property, and this proves Theorem H.

PROOF OF THEOREM I. Let k be a field of characteristic $p > 0$ and G an Abelian group.

(i) $\Rightarrow$ (ii). Suppose that $\mathfrak{g}$ has the fAR property. Let H be a finitely generated subgroup of G and x an element of G such that xH has infinite p-height. Then

$$1 - x \in \bigcap_{n=1}^{\infty} (\bar{\mathfrak{h}} + \mathfrak{g}^n)$$

(see the proof of [7], corollary 3.2) and hence by Theorem G there exists a in $\mathfrak{g}$ such that $(1-x)(1-a) \in \bar{\mathfrak{h}}$. If φ: $kG \to k(G/H)$ is the canonical homomorphism then $(1 - xH)(1 - \varphi(a)) = 0$ and, hence, xH has finite order n and n is not divisible by p. Thus (ii) holds.

(ii) ⇒ (iii). Suppose that (ii) holds. Let H be any finitely generated subgroup of G and $Q = G/H^p$. Then $\bar{H} = H/H^p$ is finite and $\bar{H} \cap (\bigcap_{n=1}^{\infty} Q^{p^n}) = 1$. Thus there exists a positive integer n such that $\bar{H} \cap Q^{p^n} = 1$, so that $H \cap G^{p^n} \leqq H^p$.

(iii) ⇒ (i). Suppose that (iii) holds. Note that for every finitely generated subgroup H and positive integer m there exists a positive integer t such that

$$H \cap G^{p^t} \leqq H^{p^m}.$$

This follows by repeated use of (iii) applied to the finitely generated subgroups $H, H^p, H^{p^2}, \cdots, H^{p^{m-1}}$. Let E be a finitely generated ideal of kG. Let X be the finitely generated subgroup of G generated by the supports of the elements in a finite generating set for E. Since X is finitely generated Abelian it follows that $\mathfrak{x}$ has the AR property in kX and so there exists a positive integer m such that $(E \cap kX) \cap \mathfrak{x}^m \leqq (E \cap kX)\mathfrak{x}$. If U is a transversal to the cosets of X in G then $E = \bigoplus_{u \in U}(E \cap kX)u$ and $\bar{\mathfrak{x}}^m = \bigoplus_{u \in U}\mathfrak{x}^m u$ and it follows that $E \cap \bar{\mathfrak{x}}^m \leqq E\mathfrak{x}$. Let $Y = X^{p^m}$. Then $\mathfrak{y} \leqq \mathfrak{x}^m$ and so $E \cap \bar{\mathfrak{y}} \leqq E\mathfrak{x} \leqq E\mathfrak{g}$. Also we have seen above that there exists a positive integer s such that

$$X \cap G^{p^s} \leqq Y.$$

Let $B = G^{p^s}$ and C be the group G/B. Then C is an Abelian group of bounded order and by [1], theorem 6, C is a direct product of cyclic groups. By Lemma 5.3 $\mathfrak{c}$ has the fAR property. Thus there exists a positive integer v such that $E \cap \mathfrak{g}^v \leqq E\mathfrak{g} + \bar{\mathfrak{b}}$. This implies $E \cap \mathfrak{g}^v \leqq E\mathfrak{g} + E \cap \bar{\mathfrak{b}} \leqq E\mathfrak{g} + E\mathfrak{b} + \bar{\mathfrak{y}}$ by Lemma 5.2. Thus

$$E \cap \mathfrak{g}^v \leqq E\mathfrak{g} + E\mathfrak{b} + E\mathfrak{b} + E \cap \bar{\mathfrak{y}} \leqq E\mathfrak{g}.$$

It follows that $\mathfrak{g}$ has the fAR property.

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